SEM 2, 2023-24: TOPOLOGY
MID-SEMESTRAL EXAMINATION

Solve any 5. Each question carries 5 marks. Total: 25 marks. Time: 3 hrs. You may apply any results proven in class without proofs, but clearly state the result you are using. If a question specifically asks for the proof of a result covered in class, you must provide a detailed proof.

(1) Show that a topological space X is Hausdorff if and only if the diagonal $\Delta_X := \{(x,x) \in X \times X | x \in X\}$ is a closed subspace of $X \times X$.

(2) For each natural number $n \geq 1$, let $f_n : [0,1] \to \mathbb{R}$ be the function defined by $x \mapsto x^n$.
  (a) Show that $(f_n(x))_n$ converges for each $x \in [0,1]$.
  (b) Show that $(f_n)_n$ does not converge uniformly.
  (Recall a sequence of functions f_n converges to a function f uniformly if given $\epsilon > 0$ there exists an integer N such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and for all x).

(3) Let $\mathbb{R}^\infty \subset \mathbb{R}^\omega$ denote the set of all real valued sequences that are eventually zero, i.e, those sequences $(x_n)_{n\in\mathbb{N}}$ such that $x_n = 0$ for all but finitely many values of n. Compute the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in **any two** of the following topologies on $\mathbb{R}^\omega$ and justify your answer: (a) Product topology, (b) uniform topology, (c) box topology.

(4) Let (X,d) be a metric space.
  (a) Show that $d : X \times X \to \mathbb{R}$ is continuous (for the product of the metric topology on $X \times X$).
  (b) Show that given any topology $\mathscr{T}$ on X such that $d : X \times X \to \mathbb{R}$ is continuous for the product topology on $X \times X$, $\mathscr{T}$ is finer than the metric topology.

(5) Let $f : X \to Y$ be a function between topological spaces X and Y. Let A, B be closed subspaces of X such that $X = A \cup B$ and $f|_A : A \to Y$ and $f|_B : B \to Y$ are continuous functions. Then show that $f : X \to Y$ is continuous.

(6) The "line with two origins" is defined as $X = (\mathbb{R} \setminus \{0\}) \cup \{0^+, 0^-\}$ with the topology consisting of open sets in $\mathbb{R} \setminus \{0\}$ and sets of the form $U^+ := (U \setminus \{0\}) \cup \{0^+\}$ and $U^- := (U \setminus \{0\}) \cup \{0^-\}$for $U \subset \mathbb{R}$ open and containing 0.
  (a) Show that this is indeed a topology on X
  (b) Prove that X is not metrizable.
  HINT: Prove X is not Hausdorff.

(7) Suppose X is a topological space and $A \subset B \subset X$. Show that A is dense in X if and only if A is dense in B (in the subspace topology) and B is dense in X. (Recall a subset $Z \subset Y$ of a topological space Y is dense in Y if the closure of Z is Y).

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